Naive Calculus

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Introduction. Does this 'logic' look familiar to you?

$$\frac{d}{dx}\left(x^2 \sin x\right) = 2x \cos x$$

Or how about this:

$$\frac{d}{dx}\sin(x^2) = \cos(2x)$$

No matter how many times we teach calculus it seems we always have a few students who consistently make mistakes of this type. Instead of using the correct rules of differentiation, they use differentiation 'rules' of the form

$$\frac{d}{dx}\left(f\left(x\right) \cdot g\left(x\right)\right) = f'\left(x\right) \cdot g'\left(x\right), \tag{1}$$

or

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g'(x)}, \tag{2}$$

or

$$\frac{d}{dx}\left(f(x) \circ g(x)\right) = f'(x) \circ g'(x). \tag{3}$$

In their Classroom Note [1], Maharam and Shaughnessy remark that some students will "go to their graves" believing these erroneous rules are true. Sadly, they were probably right, and (1), (2), and (3) will no doubt be with us for as long as calculus is taught.

Despite this, or perhaps because of it, we believe that equations (1), (2), and (3) are worthy of terminology of their own. We will refer to equation (1) as the *naive product rule*, equation (2) as the *naive quotient rule*, and equation (3) as the *naive chain rule*. A product $f(x) \cdot g(x)$ will be called *naively differentiable* if (1) holds; similarly, a naively differentiable quotient $f(x)/g(x)$ will be one that satisfies (2), and a naively differentiable composition $(f \circ g)(x)$ will be one that satisfies (3). Finally, a student who believes that (1), (2), or (3) holds in general will simply be called *naive*.

The naive product, quotient, and chain rules are part of the *naive calculus*, which we define as the set of all mistakes made by calculus students. These naive differentiation rules are false in general, but naively differentiable products, quotients, and compositions *do* exist. (A trivial example is $f(x) \equiv 0$ along with any non-constant differentiable g.) Maharam and Shaughnessy characterize in [1] the continuously differentiable functions satisfying the naive product rule, while Parris *et al* do the same for the naive quotient rule in [2]. A summary of their results is:

Theorem 1 (Maharam and Shaughnessy [1], Parris *et al* [2])**.** Suppose f and g are continuously differentiable and non-constant on an interval I. Then:

1. f and g satisfy the naive product rule (1) on I if and only if

$$f(x) = C \exp\left(\int \frac{g'(x)}{g'(x) - g(x)}\,dx\right) \quad \text{on } I$$

   for some non-zero constant C.

2. If in addition g and g' are non-zero on I, then f and g satisfy the naive quotient rule (2) on I if and only if f is of the form

$$f(x) = w(x) \cdot g(x)$$

   where w and g satisfy the naive product rule (1) on I.

Theorem 1 characterizes naively differentiable products and quotients, but there are still questions to be asked. For instance, are there products or quotients that are *repeatedly* naively differentiable? And what about naively differentiable compositions? Theorem 1 explicitly shows how to construct examples of functions satisfying the naive product and quotient rules. Is there a similar recipe for compositions? In this note we will investigate these questions and others. Along the way we will encounter mathematics that is interesting in its own right, including differential equations, dynamical systems, and even some combinatorics that harkens back to Euler.

Naive Product and Quotient Rule Examples. Before launching into repeated differentiation under the naive product and quotient rules, we present three examples which will play a role in what follows.

Example 1. Let $g(x) = x^r$, where $r \neq 0$. Then the non-constant continuously differentiable functions f satisfying the naive product rule with g are all of the form $f(x) = C(x-r)^{-r}$, where C is a non-zero constant (as pointed out by Maharam and Shaughnessy in their original note [1]). Now applying Theorem 1 part 2, we find that $h(x) = f(x) \cdot g(x) = Cx^r(x-r)^{-r}$ satisfies the naive quotient rule with numerator h and denominator g.

Example 2. Let $g(x) = C_2 e^{bx}$, where C_2 and b are non-zero. We invite the reader to verify that:

1. The product $f(x) \cdot g(x)$ is naively differentiable if and only if $f(x) =$

$C_1 e^{ax}$, where a is non-zero and satisfies

$$ab = a + b,$$

2. The quotient $f(x)/g(x)$ is naively differentiable if and only if $f(x) = C_1 e^{ax}$ where a is non-zero and satisfies

$$a/b = a - b.$$

Note that by the first part of Example 2, *no* non-trivial function f satisfies the naive product rule with $g(x) = Ce^x$, a fact which will play a role in our characterization of the "repeated naive differentiability" of products and quotients in Theorem 2 below.

Example 3. Consider $g(x) = Ae^{bx} + B$, where A, b, and B are non-zero. It is not difficult to verify that if a non-constant f satisfies the naive product rule with g, then either

1. $b \neq 1$ and $f(x) = C\left(A(b-1)e^{bx} - B\right)^{1/(b-1)}$, where C is a non-zero constant, or

2. $b = 1$ and $f(x) = Ce^{-(A/B)e^x}$, where C is a non-zero constant.

Repeated Naive Differentiation. We will say that a product $f(x) \cdot g(x)$ is *n-times naively differentiable* if

$$\frac{d^k}{dx^k}\left(f(x) \cdot g(x)\right) = f^{(k)}(x) \cdot g^{(k)}(x) \tag{4}$$

for *all* $k = 1, 2, \cdots, n$. Similarly, we say that a quotient $f(x)/g(x)$ is n-times naively differentiable if

$$\frac{d^k}{dx^k}\left(\frac{f(x)}{g(x)}\right) = \frac{f^{(k)}(x)}{g^{(k)}(x)}, \tag{5}$$

for all $k = 1, 2, \cdots, n$. It is not hard to see that a product or quotient is n-times naively differentiable if and only if it and its derivatives up to order $n-1$ are all naively differentiable. In Example 2 we saw that if a and b are non-zero, then the pair $f(x) = C_1 e^{ax}$ and $g(x) = C_2 e^{bx}$ satisfy the naive product rule if and only if $ab = a+b$, while they satisfy the naive quotient rule if and only if $a/b = a-b$. Moreover, since the derivatives of these functions take the same form, these pairs satisfy the naive product and quotient rules, respectively, for *all* orders. In Example 3, the $b = 2$ case shows that the pair $g(x) = Ae^{2x} + B$ and $f(x) = C\left(Ae^{2x} - B\right)$ satisfy the naive product rule. Again, since the derivatives of these functions are included in Example 2, these pairs also satisfy the naive product for all orders. What other products or quotients are n-times naively differentiable for some $n \geq 2$? The answer is essentially none, at least among the very 'nice' functions.

Theorem 2. Suppose f and g are non-constant analytic functions. Then:

1. The product $f(x) \cdot g(x)$ is twice naively differentiable if and only if either

   (a) $f(x) = Ae^{ax}$ and $g(x) = Be^{bx}$, where A and B are non-zero constants, and $a, b \notin \{0, 1\}$ satisfy $ab = a + b$.

   (b) $f(x) = C(Ae^{2x} - B)$ and $g(x) = Ae^{2x} + B$, where A, B and C are constants, with A and C non-zero.

   Moreover, in both these cases, the product $f \cdot g$ is infinitely naively differentiable.

2. The quotient $\frac{f(x)}{g(x)}$ is twice naively differentiable if and only if $f(x) = w(x)g(x)$, where w and g are a twice naively differentiable product as in 1. Moreover, in this case the quotient $\frac{f}{g}$ is infinitely naively differentiable.

Proof. We have already seen that if f and g are of the given forms, then they are naively differentiable to all orders. So we need only prove that twice-naively-differentiable products or quotients must be composed of functions of the given forms. We will do the proof for products, leaving the analogous proof for quotients to the reader.

Suppose that f and g are non-constant analytic functions such that the product $f(x) \cdot g(x)$ is twice naively differentiable. Then (4) holds for $k = 1$ and 2, and it follows that the first part of Theorem 1 applies to the pair f' and g' as well as to the pair f and g. Therefore in addition to

$$f(x) = C_1 \exp\left(\int \frac{g'(x)}{g'(x) - g(x)}\, dx\right), \tag{6}$$

we also have

$$f'(x) = C_2 \exp\left(\int \frac{g''(x)}{g''(x) - g'(x)}\, dx\right). \tag{7}$$

But differentiating equation (6) gives another expression for f', and after equating it with (7), we get

$$C_1 \exp\left(\int \frac{g'(x)}{g'(x) - g(x)}\, dx\right) \cdot \frac{g'(x)}{g'(x) - g(x)} = C_2 \exp\left(\int \frac{g''(x)}{g''(x) - g'(x)}\, dx\right). \tag{8}$$

Taking the natural logarithm of both sides of (8) and differentiating the result (non-naively) yields

$$\frac{g''}{g'' - g'} = \frac{g'}{g' - g} + \frac{(g')^2 - g''g}{(g' - g)\, g'}. \tag{9}$$

This is a fairly intimidating differential equation for g, but, thankfully, clearing out the denominators of (9) and factoring the result by grouping yields

$$\left((g')^2 - g''g\right)\left(g'' - 2g'\right) = 0, \tag{10}$$

which is equivalent to (9) if none of g', $g' - g$ or $g'' - g'$ is identically zero on any interval. But if g' were identically zero on an interval, then g would be a constant, contradicting our assumptions. If $g' - g$ were zero on an interval, then $g(x) = Ce^x$, which would imply that f is a constant, again in contradiction to our assumptions. Finally, if $g'' - g'$ were zero on an interval, then $g(x) = Ce^x + B$ (where $C \neq 0$ by our assumptions), and so $g'(x) = Ce^x$. Since $f' \cdot g'$ is naively differentiable, we must have $f'(x) = D$, where D is a constant, and thus $f(x) = Dx + E$. Now $f \cdot g$ is also naively differentiable, which easily implies that $D = E = 0$, again contradicting our assumptions on f. Therefore, equations (9) and (10) are equivalent, and any g satisfying our assumptions must also satisfy (10).

Now, one of the factors on the left side of (10) must be identically zero. If it is the first, then we find that $g(x) = Be^{bx}$ and $f(x) = Ae^{ax}$, where $A, B \neq 0$, $a, b \notin \{0, 1\}$, and $ab = a + b$. If it is the second, we find that $g(x) = Ae^{2x} + B$, $f(x) = C(Ae^{2x} - B)$, where A, B and C are constants, with A and C non-zero. This completes the proof. $\square$

Theorem 2 shows that the requirement that a product or quotient be twice naively differentiable is quite stringent: it forces a very specific form on the functions involved. Actually, this is not surprising in light of Theorem 1 above. In the case of products, the first part of the theorem states that the set of functions f that are naively differentiable with a given g forms a one-parameter family. But a twice naively differentiable function is of course naively differentiable, so not only must this one-parameter family *already* contain f', but f' must *in addition* be naively differentiable with g'. Having all this happen at once would seem to be an unlikely convergence of coincidences for 'most' functions g.

Instead of imposing the stringent requirement of naive differentiability *up to* a certain order, we could simply require naive differentiability *at* a given order; i.e. require a pair of functions to satisfy (4) or (5) for some *single* value of $k > 1$. This is certainly possible, and, in the case of the product rule, leads (via Leibniz's formula for the n^{th} derivative of a product) to an n^{th}-order linear differential equation with non-constant coefficients. Such equations have solutions if the coefficients are continuous (see [3], for instance), and so it is possible to generate examples. For instance, we leave it to the reader to verify that if g is a monic polynomial of degree $n \geq 1$, then $f(x) = p(x) / (g(x) - n!)$ satisfies (4) for $k = n$, where $p(x)$ is any polynomial of degree $n-1$ or less. This can be thought of as a generalization of the $r = 1$ case of Example 1.

Before turning our attention to the naive chain rule, we leave the reader with two questions: attempting to generalize Theorem 2, if $n > 1$ does naive differentiability at orders n and $n+1$ imply that f and g are the exponentials of Example 2? More generally, does naive differentiability at orders n and $n + m$ imply the same thing?

Naive Chain Rule. Naively differentiable products and quotients are relatively easy to categorize, even for some higher orders of differentiation. However, as we will see, the situation is more difficult to grasp in the case of the naive chain rule.

In order for the naive chain rule $(f \circ g)' = f' \circ g'$ to hold, we obviously must have

$$f'(g(x))\, g'(x) = f'(g'(x)). \tag{11}$$

We will investigate solutions of (11) by either fixing the outer function f and solving for g, or conversely, by fixing the inner function g and solving for f. It turns out that these two approaches lead to very different solution techniques. For a fixed outer function f, equation (11) is an ordinary differential equation for g. On the other hand, for a fixed inner function g, (11) can be considered as a functional equation for f'. We think this approach is particularly interesting, because it leads in some cases to solutions f' whose properties are determined by the dynamics of an iterated function related to g.

Fixing f and Solving for g. Note first that if f is a constant function, then (11) holds for *any* differentiable g. We will refer to this as the *trivial case* of the naive chain rule. For certain non-trivial f, we can still solve equation (11) for g. In fact this solution technique involves similar considerations that were already addressed in the product and quotient rule section. As such, we highlight only one example.

Example 4. Let $f(x) = e^x$, for $x \geq 0$. Then equation (11) becomes $e^{g(x)} g'(x) = e^{g'(x)}$, or

$$\frac{e^{g'(x)}}{g'(x)} = e^{g(x)}. \tag{12}$$

This equation does not seem to be explicitly solvable for g. Nevertheless, we can prove that a solution exists, and thus that there exists a function g satisfying the naive chain rule with $f(x) = e^x$. Let $k(z) = e^z/z$, and note that the restriction of k to $[1,\infty)$ is invertible, mapping $[1,\infty)$ monotonically onto $[e,\infty)$. Denote this restriction by $\hat{k}$, and consider the initial value problem

$$\begin{aligned} g'(x) &= \hat{k}^{-1}\left(e^{g(x)}\right) \\ g(0) &= g_0, \end{aligned} \tag{13}$$

where $g_0 > 1$.

The right side of initial value problem (13) is Lipschitz continuous on $[g_0,\infty)$ for all $g_0 > 1$. Therefore it has a unique solution $g(x)$ (see [4], Theorem 58.5, for instance). Moreover, this solution will be defined for all $x \in [0,\infty)$. This solution g will satisfy (12), and therefore will satisfy the naive chain rule with outer function $f(x) = e^x$. ■

The approach in Example 4 can be generalized to some extent. However, we will leave this line of inquiry to the interested reader, and turn instead to a completely different approach for generating examples of the naive chain rule.

Fixing g and solving for f. In the previous section we fixed an outer function f, and found inner functions g satisfying the naive chain rule by solving an ODE. In this section, we will reverse things by fixing the inner function g, and finding corresponding outer functions f.

We begin by assuming that the given inner function g satisfies an autonomous first-order differential equation

$$g' = p\left(g\right), \tag{14}$$

where p is some suitably nice function (we will discuss the specific niceties required of p soon). While (14) may initially seem restrictive, it is satisfied for any function that is monotone on an interval. So, using (14) we see that (11) becomes $f'\left(p\left(g\right)\right) = f'\left(g\right) \cdot p\left(g\right)$, or more succinctly,

$$h \circ p = h \cdot p, \tag{15}$$

where $h = f'$. Note that equation (15) itself may be thought of as a form of "naive composition." If in addition p maps the range of g into itself, then the possibility of iterating p in equation (15) arises. Let $y = g\left(x\right)$, and let $p^{(n)}\left(y\right)$ denote the n^{th} iterate of p (i.e. $p^{(2)}\left(y\right) = p\left(p\left(y\right)\right)$, and in general $p^{(n+1)}\left(y\right) = p\left(p^{(n)}\left(y\right)\right)$). Then applying (15) we have

$$\begin{aligned} h\left(p^{(2)}\left(y\right)\right) &= h\left(p\left(y\right)\right) \cdot p^{(2)}\left(y\right) \\ &= h\left(y\right) \cdot p\left(y\right) \cdot p^{(2)}\left(y\right) \end{aligned}$$

and inductively we get

$$h\left(p^{(n)}\left(y\right)\right) = h\left(y\right) \cdot \prod_{i=1}^{n} p^{(i)}\left(y\right) \tag{16}$$

for all positive integers n. Provided that p maps the range of g into itself, equation (16) holds if and only if equation (15) does. In certain cases equation (16) can be used to construct outer functions f satisfying the naive chain rule with a given inner function g. In other cases, it can be used to show that *no* non-constant f satisfies the naive chain rule with a given g. Our first example is of this type.

Example 5. Let $p(y) = 4y(1-y)$ for $y \in [0,1]$. Some readers may recognize p as the *logistic map*. Students usually make its acquaintance as a differential equation modeling population growth. Later they may encounter it again in an introductory course on dynamical systems, where it is one of the simplest discrete dynamical systems that is *chaotic* (see [5], Definition 8.5 and Example 8.9). Interestingly, in this example it will appear in both these guises.

Using separation of variables and a partial fraction decomposition to solve $g' = p(g)$ gives inner functions of the form

$$g(x) = \frac{1}{1+ce^{-4x}}, \tag{17}$$

where c is an arbitrary constant. We seek outer functions f satisfying the naive chain rule with g. Since p maps $[0,1]$ into itself, any such f must satisfy (16) for all positive integers n, with $h = f'$.

Note that if $y_0 \in [0,1]$ is periodic under iteration by p with period n (i.e. $p^{(n)}(y_0) = y_0$), then by (16) we have

$$h(y_0) = h(y_0) \cdot \prod_{i=1}^{n} y_i, \tag{18}$$

where $y_i = p^{(i)}(y_0)$ are the iterates of y_0. This implies that either $h(y_0) = 0$ or $\prod_{i=1}^{n} y_i = 1$. If y_0 is in the interior of $[0,1]$, then the latter case is impossible. Hence we conclude that if $y_0 \in (0,1)$ is periodic under p, then $h(y_0) = 0$. But as mentioned above, p is chaotic, and so its periodic points are dense in $[0,1]$. Therefore if h is continuous, then we must have $h(y) = 0$ for all $y \in [0,1]$. But $h = f'$. We therefore conclude that there is no non-constant continuously differentiable f that satisfies the naive chain rule with g. ■

Extracting the essentials out of Example 5, we get the following theorem.

Theorem 3. Let $g' = p(g)$ where p is continuous and maps $[0,1]$ into itself. If the periodic points of p are dense in $[0,1]$, then the only continuously differentiable functions f satisfying the naive chain rule with g as the inner function are the constant functions $f(x) = C$.

Our next three examples use Theorem 3 to show that, for some very familiar inner functions g, there is no non-trivial outer function f satisfying the naive chain rule.

Example 6. Let $g(x) = e^x$. The corresponding p is the identity function $p(y) = y$. *Every* point is periodic for p (with period 1). Hence by Theorem 3 the only continuously differentiable f satisfying the naive chain rule with e^x are the constant functions. ■

While the result of Example 6 can be derived easily without the aid of Theorem 3, this does not appear to be the case with our next example.

Example 7. Let $g(x) = \sin x$ and restrict the domain of g to $[0, \pi/2]$. On this interval, g satisfies the differential equation $g' = p(g)$, where

$$p(y) = \sqrt{1 - y^2} \tag{19}$$

Note that p maps $[0, 1]$ into itself, and that p is its own inverse. Hence $p^{(2)}(y) = y$ for all $y \in [0, 1]$, and again every point in $[0, 1]$ is periodic under p. Therefore by Theorem 3 the only continuously differentiable functions f satisfying the naive chain rule with $g(x) = \sin x$ are the constant functions.

■

Example 8. $g(x) = \cos x$ satisfies the same differential equation (19) as $\sin x$, but on a different interval. Therefore the same conclusion holds: the only continuously differentiable functions f satisfying the naive chain rule with $g(x) = \cos x$ are the constant functions. ■

As a special case of Theorem 3 we can see that whenever g satisfies (14) with a p that is self-invertible on $[0,1]$, then no non-constant continuously differentiable f will satisfy the naive chain rule with g. Already it is clear that the set of functions g for which there is *no* non-trivial f satisfying (3) is much larger in the case of the naive chain rule than it is for the corresponding cases of the naive product and quotient rules.

Now we turn to situations in which we *can* find non-trivial f and g satisfying the naive chain rule.

Example 9. Consider $p(y) = y^{\alpha}$, where $\alpha \in (0,1)$ and $y \in (0,\infty)$. By using separation of variables, we find that the corresponding g satisfying $g' = p(g)$ is

$$g(x) = (x(1-\alpha) + c)^{1/(1-\alpha)}. \tag{20}$$

To find a function f satisfying the naive chain rule with g as the inner function, we use equation (16). For our choice of p, we have $p^{(n)}(y) = y^{\alpha^n}$. Thus equation (16) becomes

$$\begin{aligned} h\left(y^{\alpha^n}\right) &= h(y) \prod_{i=1}^{n} y^{\alpha^i} \\ &= h(y) \cdot y^{\left(\alpha - \alpha^{n+1}\right)/(1-\alpha)} \end{aligned}$$

Letting $n \to \infty$, we have $\alpha^n \to 0$ and $y^{\alpha^n} \to 1$ for all $y \in (0,\infty)$. Therefore we get $h(1) = h(y) \cdot y^{\alpha/(1-\alpha)}$, or

$$h(y) = c\, y^{-\alpha/(1-\alpha)} \tag{21}$$

where $c = h(1)$ is an arbitrary constant. Remembering that $h = f'$, we see that

$$f(y) = \begin{cases} c_2\, y^{(1-2\alpha)/(1-\alpha)} + c_3 & \text{if } \alpha \neq 1/2 \\ \\ c_2 \ln y + c_3 & \text{if } \alpha = 1/2 \end{cases} \tag{22}$$

■

Example 10. Let $p(y) = qy + (1-q)$, where $q \in (0,1)$ and $y \in [0,\infty)$. The corresponding inner function is $g(x) = 1 - q^{-1} + ce^{qx}$. To apply equation (16), note that $p^{(n)}(y) = q^n y + (1-q^n)$. Hence

$$h\left(p^{(n)}(y)\right) = h(y) \prod_{i=1}^{n} \left(q^i y + \left(1 - q^i\right)\right).$$

Letting $n \to \infty$, we have $p^{(n)}(y) \to 1$, and so

$$h(1) = h(y) \prod_{i=1}^{\infty} \left(1 + q^i (y-1)\right). \tag{23}$$

The infinite product in (23) is finite and non-zero, as is easily seen from the following Lemma (see [6], Section 3.7, Theorem 5):

Lemma 4. Suppose that b_i, $i \geq 1$ is a sequence of real numbers satisfying $\sum_{i=1}^{\infty} |b_i| < \infty$. Then $\prod_{i=1}^{\infty} (1 + b_i)$ converges. Moreover, if none of the b_i are equal to -1, then the infinite product is non-zero.

Now that we know the product in (23) converges and is non-zero, we solve for $h = f'$ to obtain

$$f'(y) = c\prod_{i=1}^{\infty}\left(1 + q^i (y-1)\right)^{-1}, \tag{24}$$

where $c = h(1)$ is an arbitrary constant. Therefore any antiderivative of (24) satisfies the naive chain rule with the inner function $g(x) = 1 + q^{-1} + ce^{qx}$.

The function f' in this example has connections to *q-series*, a class of functions tracing its lineage back to Euler (see Chapter 16 of [7], for instance). The q-series $(a;q)_\infty$ is defined by

$$(a;q)_\infty = \prod_{i=0}^{\infty}\left(1 - aq^i\right). \tag{25}$$

In terms of q-series, our function f' may be written

$$f'(y) = \frac{c}{(-zq;q)_\infty}, \tag{26}$$

where $z = y - 1$. Euler discovered fascinating relationships between q-series and combinatorics. For instance, note that the denominator in (26) is

$$(-zq;q)_\infty = (1 + zq)\left(1 + zq^2\right)\left(1 + zq^3\right)\cdots. \tag{27}$$

Think about formally multiplying out this infinite product: the result will be an infinite sum of terms of the form $z^k q^n$. How many times will $z^k q^n$

appear in the expansion? This will be the number of ways of picking k factors of the form zq^{i_j}, where the i_j's are distinct positive integers, and $i_1 + i_2 + \cdots + i_k = n$. It follows that the coefficient of $z^k q^n$ in the series expansion of (27) is the number of partitions of the integer n into exactly k parts, all of which are distinct. For example, the coefficient of $z^3 q^9$ is 3, corresponding to the partitions $1+2+6$, $1+3+5$, and $2+3+4$. For more information on q-series and partitions, see [8].

■

Extracting the essentials out of Examples 9 and 10, we have

Theorem 5. Let $p : [0,\infty) \to [0,\infty)$ be continuously differentiable, with $p(1) = 1$ and $|p'(1)| < 1$. Let g be the solution of the initial value problem

$$\begin{aligned} g' &= p(g) \\ g(x_0) &= 1, \end{aligned} \tag{28}$$

and let f satisfy

$$f'(x) = c \left(\prod_{i=1}^{\infty} p^{(i)}(x) \right)^{-1}, \tag{29}$$

where c is any constant. Then f and g satisfy the naive chain rule (3).

Proof. We need only show that the infinite product in (29) converges and is non-zero, since the remaining logic remains unchanged from Example 10.

Since p is continuously differentiable and $|p'(1)| < 1$, there exists a neighborhood N containing 1, and a positive constant $A < 1$, such that $|p'(x)| < A$ for all $x \in N$. Therefore by the mean value theorem, for any $x \in N$ we have

$$\begin{aligned}
|p(x) - 1| &= |p(x) - p(1)| \\
&= |p'(\hat{x})| \, |x - 1| \quad (\text{for some } \hat{x} \in N) \\
&< A \, |x - 1| \, . \qquad (30)
\end{aligned}$$

Therefore $p(x)$ is closer to 1 than x is, and in particular, p maps N into itself. Now iterating the inequality (30) yields $\left|p^{(n)}(x) - 1\right| < A^n |x - 1|$ for all positive integers n. Lemma 4 now guarantees the convergence of the infinite product in (29) to a finite non-zero value. □

The conditions of Theorem 5 are by no means necessary for the existence of pairs of functions satisfying the naive chain rule, as our last example shows.

Example 11. Take $p(y) = \alpha y$, where $\alpha \in (1, \infty)$ and $y \geq 1$. This corresponds to the inner function $g(x) = e^{\alpha x}$. Then $p^{(n)}(y) = \alpha^n y$, and so by (16) we have

$$\begin{aligned}
h(\alpha^n y) &= h(y) \cdot \prod_{i=1}^{n} \alpha^i y \\
&= h(y) \, y^n \alpha^{n(1+n)/2}.
\end{aligned}$$

Letting $y = 1$ and $x = \alpha^n$ yields

$$h(x) = cx^{(1+\log_\alpha x)/2}, \tag{31}$$

where $c = h(1)$ is an arbitrary constant. Strictly speaking we have only shown that (31) holds for values of x such that $\log_\alpha x$ is a positive integer. However, we can directly verify that in fact

$$f(x) = \int x^{(1+\log_\alpha x)/2} dx$$

satisfies the naive chain rule with inner function $g(x) = e^{\alpha x}$ for all positive x and *all* $\alpha > 0$, $\alpha \neq 1$. ■

Table 1 below collects various pairs of functions that satisfy one of the naive differentiation rules we have discussed. We invite you to have some 'naive' fun by generating new examples for yourself!

Naive Product Rule: $(\mathbf{fg})' = \mathbf{f}'\mathbf{g}'$	
$f(x) = e^{ax}$	$g(x) = e^{ax/(a-1)}$
$f(x) = e^x - 1/c$	$g(x) = e^{ce^x}$
$f(x) = \sin^2 x$	$g(x) = \frac{e^{4x/5}}{(\sin x - 2\cos x)^{2/5}}$
$f(x) = e^{ax}\sin^b x$	$g(x) = \left(\frac{e^{\left(a^2-a+b^2\right)x}}{[(a-1)\sin x + b\cos x]^b}\right)^{\frac{1}{(a-1)^2+b^2}}$
Naive Quotient Rule: $(\mathbf{f}/\mathbf{g})' = \mathbf{f}'/\mathbf{g}'$	
$f(x) = e^{ax}$	$g(x) = e^{\frac{a\pm\sqrt{a^2-4a}}{2}x}$
$f(x) = (e^x - 1/c)\, e^{ce^x}$	$g(x) = e^{ce^x}$
$f(x) = \frac{e^{4x/5}\sin^2 x}{(\sin x - 2\cos x)^{2/5}}$	$g(x) = \frac{e^{4x/5}}{(\sin x - 2\cos x)^{2/5}}$
Naive Chain Rule: $(\mathbf{f}\circ\mathbf{g})' = \mathbf{f}'\circ\mathbf{g}'$	
$f(x) = \ln x$	$g(x) = x^2/4$
$f(x) = x^{2-1/\beta}$	$g(x) = (\beta x)^{1/\beta}$ $(\beta \in (0,1), \beta \neq 1/2)$
$f'(x) = x^{(1+\log_\alpha x)/2}$	$g(x) = e^{\alpha x}$ $(\alpha > 0, \alpha \neq 1)$
$f'(x) = \prod_{i=1}^{\infty}\left(1 + q^i(x-1)\right)^{-1}$	$g(x) = 1 - q^{-1} + e^{qx}$ $(q \in (0,1))$

Table 1: Various Naively Differentiable Function Pairs

References

[1] Lewis Maharam and Edward Shaughnessy, When Does $(fg)' = f'g'$?, *The Two-Year College Mathematics Journal*, Vol. 7, No. 1, Feb. 1976, 38-39.

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Referee's Appendix.

This section consists of notes and comments that the authors felt might be of help to the referees of this paper.

Note 1: Abstract of this paper. A product (or quotient or composition) of functions is said to be *naively differentiable* if the derivative of the product (or quotient or composition) is the product (or quotient or composition) of the derivatives. Naively differentiable products and quotients have been characterized in [1] and [2]. In this note we characterize products and quotients that are *n-times naively differentiable*, meaning that the product or quotient and all its derivatives up to order $n-1$ are naively differentiable. We also study pairs of functions satisfying the *naive chain rule* $(f \circ g)' = f' \circ g'$. Two distinct approaches are used to study the naive chain rule. First, by fixing the outer function f, inner functions g satisfying the naive chain rule with f can be found by solving an ordinary differential equation. Second, by fixing the inner function g, outer functions f satisfying the naive chain rule with g can be found by iterating a functional equation. Lastly, we give several examples of function pairs that are naively differentiable.

Note 2: The statement of Theorem 1. We have simplified the statement of the theorem as much as possible, while preserving its main ideas and applicability to our context in this paper. In particular, in part 1 of the theorem, it is implicit that $g'(x) - g(x) \neq 0$. To see this, in the proof of necessity (the "$\Rightarrow$" direction), if f and g satisfy the naive product rule and either one is equal to Ce^x, then the other one has to be a constant, in contradiction to the assumption that f and g are both non-constant. Hence $g'(x) - g(x) \neq 0$ in particular. In the proof of sufficiency (the "$\Leftarrow$" direction), it is implicit in the assumed form of f that $g'(x) - g(x) \neq 0$. Thus, we believe that the theorem is correct as stated, with no need for qualifications.

Also, in part 2, we have stated a result slightly less general than that of Parris *et al* in [2], who made no assumption of continuous differentiability. Still, the result as stated is sufficient for our purposes.

Note 3: A product (or quotient) is n-times naively differentiable if and only if it and all its derivatives up to order $n-1$ are naively differentiable. We do the proof for naively differentiable products. We need to show that

The product $f(x) \cdot g(x)$ is n-times naively differentiable

$$\Leftrightarrow$$

$$\frac{d}{dx}\left(f^{(k)}(x) \cdot g^{(k)}(x)\right) = f^{(k+1)}(x) \cdot g^{(k+1)}(x) \text{ for all } k = 0, 1, \cdots n-1 \quad (32)$$

Proof. We will do the proof for naively differentiable products. The proof for naively differentiable quotients is similar.

($\Rightarrow$) We argue by induction. The $n = 1$ case is simply the definition of the naive differentiability of a product. The inductive assumption is that for some $m \geq 1$ equation (32) holds for $n = m$ whenever the product $f \cdot g$ is m-times naively differentiable. Now assume that $f \cdot g$ is $m+1$-times naively differentiable. Then *a fortiori*

$$\frac{d^{m+1}}{dx^{m+1}}(f(x) \cdot g(x)) = f^{(m+1)}(x) \cdot g^{(m+1)}(x),$$

and because $f \cdot g$ is m-times naively differentiable, equation (32) holds by the inductive assumption. Putting these facts together,

$$\begin{aligned} f^{(m+1)}(x) \cdot g^{(m+1)}(x) &= \frac{d^{m+1}}{dx^{m+1}}(f(x) \cdot g(x)) \\ &= \frac{d}{dx}\frac{d^m}{dx^m}(f(x) \cdot g(x)) \\ &= \frac{d}{dx}\left(f^{(m)}(x) \cdot g^{(m)}(x)\right). \end{aligned}$$

Hence (32) holds for $n = m + 1$, completing the induction.

($\Leftarrow$) Again we use induction, with the $n = 1$ case being the definition of the naive differentiability of a product. The inductive assumption is that for some $m \geq 1$, if equation (32) holds for $n = m$, then $f \cdot g$ is m-times naively differentiable. Now assume that (32) holds for $n = m + 1$. Then it certainly holds for $n = m$, and so $f \cdot g$ is m-times naively differentiable. Using this along with the $n = m + 1$ case of (32) yields

$$\begin{aligned} f^{(m+1)}(x) \cdot g^{(m+1)}(x) &= \frac{d}{dx}\left(f^{(m)}(x) \cdot g^{(m)}(x)\right) \\ &= \frac{d}{dx}\frac{d^m}{dx^m}\left(f(x) \cdot g(x)\right) \\ &= \frac{d^{m+1}}{dx^{m+1}}\left(f(x) \cdot g(x)\right). \end{aligned}$$

This and the fact that $f \cdot g$ is m-times naively differentiable implies that $f \cdot g$ is $m + 1$-times naively differentiable, completing the induction.

Note 4: On the proof of Theorem 2. We claim that, in the context of the Theorem, if $(g')^2 - g''g = 0$ on an interval, then $g(x) = Ce^{bx}$ for some constants C and b. Indeed, if $(g')^2 - g''g = 0$ on an interval, then $g'/g = g''/g'$. Therefore

$$\frac{d}{dx}\ln(g) = \frac{d}{dx}\ln(g')$$

on some interval, implying that $g' = bg$ on that interval, where b is non-zero. Hence $g(x) = Ce^{bx}$ as claimed.

Also, if $g'' - 2g' = 0$ on an interval, then clearly $g(x) = Ce^{2x} + B$.

Note 5: The proof of Theorem 2 for quotients. To begin we consider functions $f(x)$ and $g(x)$ which satisfy

$$\left(\frac{f}{g}\right)' = \frac{f'}{g'}$$

This question has appeared, with the solution, in [2]. To summarize, such functions must satisfy the following:

$$\begin{aligned}
&\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} = \frac{f'}{g'} \\
\implies \quad & gf'g' - f(g')^2 = f'g^2 \\
\implies \quad & f' - \frac{(g')^2}{gg' - g^2} f = 0 \\
\implies \quad & \frac{d}{dx} f(x) = A(x) \cdot f(x), \qquad \text{where } A(x) = \frac{(g'(x))^2}{g(x)g'(x) - (g(x))^2}
\end{aligned}$$

So, given $g(x)$, we get $\mu = e^{-\int A(x)dx}$, and $f(x) = C\mu^{-1}$.

Next we ask, what quotients f/g are twice naively differentiable? We saw from Theorem 1 that, given g, $f = C_1 \exp\left(\int \frac{(g')}{g'-g}\, dx\right) g$. Similarly, using the fact that $\left(\frac{f'}{g'}\right)' = \frac{f''}{g''}$, we see that, given a g, $f' = C_2 \exp\left(\int \frac{g''}{g''-g'}\, dx\right) g'$.

We now perform the following steps: take the derivative of the expression for f, set it equal to the expression for f', and apply natural log to both sides. The result is:

$$\ln(C_2) + \int \frac{g''}{g'' - g'}\, dx + \ln(g') = \ln(C_1) + \int \frac{g'}{g' - g}\, dx + \ln\left(\frac{(g')^2}{g' - g}\right)$$

Differentiating both sides yields

$$\frac{g''}{g'' - g'} + \frac{g''}{g'} = \frac{g'}{g' - g} + \frac{g' - g}{(g')^2}\left(\frac{2g'g''(g' - g) - g'(g'' - g')}{(g' - g)^2}\right)$$

Notice that the denominators here cannot be zero (they are the same denominators we saw in the proof of the product rule portion of this theorem.) With this in hand, we see that this monstrosity simplifies (via factoring) to:

$$0 = ((g')^2 - gg'')(g'' - 2g')$$

This is the same condition we saw when discussing twice naively differentiable products. As such, either (a) $g = Be^{bx}$ $(B \neq 0, b \notin \{0, 1\})$ or (b) $g = Ae^{2x} + B$ $(A \neq 0)$, and $f = wg$, where either (a) $w = Ae^{ax}$ $(A \neq 0, a \notin \{0, 1\}, ab = a + b)$ or (b) $w = C(Ae^{2x} - B)$ $(A, C \neq 0)$ respectively.

Note 6: Let $g(x)$ be any monic polynomial of degree $n \geq 1$. We seek f such that

$$\frac{d^n}{dx^n}\left(f(x) \cdot g(x)\right) = f^{(n)}(x) \cdot g^{(n)}(x). \tag{33}$$

But $g^{(n)}(x) = n!$, so (33) is equivalent to

$$\frac{d^n}{dx^n}\left(f(x)\cdot g(x)\right) = n! f^{(n)}(x).$$

Subtracting the right side from the left, we get

$$\frac{d^n}{dx^n}\left(f(x)\cdot g(x) - n! f(x)\right) = 0.$$

Now integrating n times with respect to x gives

$$f(x)\cdot g(x) - n! f(x) = p(x),$$

where p is an arbitrary polynomial of degree $n-1$ or less. Solving this latter equation for f gives

$$f(x) = \frac{p(x)}{g(x) - n!}.$$

Note that all these equations are equivalent to each other.

Note 7: Notes on Example 4. We will justify the statement that the solution of (13) is unique, and defined for all $x \geq 0$. First we give a general proposition, which is probably well-known:

Proposition A1. Consider the initial value problem

$$\begin{aligned} \frac{dg}{dx} &= \phi(g) \\ g(x_0) &= g_0, \end{aligned} \qquad (34)$$

where ϕ is Lipschitz continuous and positive for all $x \geq g_0$. Then (34) has a unique solution. Moreover, this solution is defined for all $x \geq x_0$ if and only if

$$\int_{g_0}^{\infty} \frac{1}{\phi(g)}\, dg = \infty \tag{35}$$

Proof. The existence of a unique solution is a standard result (see [4], Theorem 58.5, for instance). Since $\phi > 0$, the solution g is strictly increasing, and hence invertible. Denoting the inverse function by x, we have from (34) that

$$\frac{dx}{dg} = \frac{1}{\phi(g)}$$

$$x(g_0) = x_0.$$

Therefore x is given by

$$x(g) = x_0 + \int_{g_0}^{g} \frac{1}{\phi(s)}\, ds \tag{36}$$

Now if $\int_{g_0}^{\infty} \frac{1}{\phi(s)}\, ds = \infty$, then from (36) x takes on all values greater than x_0 and, and so the solution g of (34) is defined for all $x \geq x_0$. On the other hand, if $\int_{g_0}^{\infty} \frac{1}{\phi(s)}\, ds = L < \infty$, then x takes on only values less than $x_0 + L$, and therefore the solution g of (34) is defined only on the interval $[x_0, x_0 + L)$. □

Now in the context the IVP (13) of Example 4, we have $\phi(g) = \hat{k}^{-1}(e^g)$.

It follows that if $z = \phi(g)$, then $g = z - \ln z$, and so

$$\begin{aligned}\int_{g_0}^{\infty} \frac{1}{\phi(g)}\, dg &= \int_{z_0}^{\infty} \frac{1}{z}\left(1 - \frac{1}{z}\right) dz \qquad (\text{since } dg = (1 - 1/z)\, dz) \\ &= \int_{z_0}^{\infty} \frac{z-1}{z^2}\, dz \\ &= \infty,\end{aligned}$$

where $z_0 > 1$. We conclude from the Proposition that the solution of (13) is defined for all $x \geq 0$.

Note 8: Proof of Theorem 3.

Proof. Let g and p satisfy the hypotheses of the Theorem. Let $y \in (0,1)$ be a periodic point of p, and let n be the period of y. If f satisfies the naive chain rule with g, then by equation (16) we must have

$$h(y) = h(y) \cdot \prod_{i=1}^{n} p^{(i)}(y), \tag{37}$$

where $h = f'$. Therefore either $h(y) = 0$ or $\prod_{i=1}^{n} p^{(i)}(y) = 1$. The latter alternative is impossible, however, since all the terms of the product are in $[0,1]$ and the last term $p^{(n)}(y) = y$ is strictly less than 1. We conclude that $h(y) = 0$ for every periodic point $y \in (0,1)$ of p. But such points are dense in $(0,1)$ by assumption, hence $f'(y) = 0$ in a dense subset of $[0,1]$. If f'

is continuous, this implies that $f'(x) = 0$ for all $x \in [0,1]$, and therefore $f(x) = C$ for some constant C. □

Note 9: Proof of Lemma 4.

Proof. A little calculus shows that there exists $\delta \in (0,1)$ such that $|\ln(1+x)| \leq 2|x|$ whenever $|x| \leq \delta$. The b_i converge to 0, so there exists $n_0 \in \mathbb{N}$ such that $|b_i| \leq \delta$ for all $i \geq n_0$. Therefore,

$$\sum_{i=n_0}^{\infty} |\ln(1+b_i)| \leq 2 \sum_{i=n_0}^{\infty} |b_i| < \infty$$

Therefore the infinite series $\sum_{i=n_0}^{\infty} \ln(1+b_i)$ converges absolutely. Exponentiating this series, we see that $\prod_{i=n_0}^{\infty} (1+b_i)$ does converge to a finite non-zero value. Hence, $\prod_{i=1}^{\infty} (1+b_i)$ converges, and moreover is non-zero provided that none of the b_i are equal to -1. □